ON SOME PROBLEMS ASSOCIATED WITH DIVISOR FUNCTIONS AND THE MÖBIUS FUNCTION

Kevin Smith

A DISSERTATION in Pure Mathematics

Submitted by Kevin Smith to the University of Exeter as a dissertation for the Degree of M.Sc in Mathematics by Research.

June, 2017

I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

ON SOME PROBLEMS ASSOCIATED WITH DIVISOR FUNCTIONS AND THE MÖBIUS FUNCTION

Kevin Smith

Dr Julio Andrade, Advisor

Contents

0.1 Overview

This thesis is about three separate problems in Analytic Number Theory, addressed separately in Chapters 1, 2 and 3. It is intended that each chapter is regarded as a miscellaneous topic; as such the chapters may be read in any order.

We begin with a brief summary of the main results of each chapter. Further details including other results, conjectures and references, are given in the foregoing introductory sections for each chapter.

0.1.1 Chapter 1. On the local maxima of divisor functions

In Chapter 1 we generalise of a result of Erdős and Hall [8] (Theorem 1.1.1 herein) on the local maxima of divisor functions. Let n be a natural number, let $k \geq 2$ and let $d_k(n)$ denote the number of ways of writing n as a product of k factors. We consider the problem of determining non-trivial bounds for the quantity $E_k(x,h)$ in the asymptotic relation

$$\sum_{n\leq x} \max\{d_k(n), d_k(n+h)\} = 2\sum_{n\leq x} d_k(n) + E_k(x,h) \tag{0.1.1}$$

when $h \neq 0$ is fixed.

The case $k = 2$ is dealt with in [8], where the authors establish that

$$E_2(x,h) = O(x(\log x)^{2(\sqrt{2}-1)}) \tag{0.1.2}$$

for fixed h. The main result of Chapter 1 is Theorem 1.1.2, in which it is established that

$$E_k(x,h) = O(x(\log x)^{2(\sqrt{k}-1)}) \tag{0.1.3}$$

with h and k fixed. The method used by Erdős and Hall to prove (0.1.2) depends crucially on a particular growth constraint on $d_k^{1/2}(p^\alpha)$, $\alpha \in \mathbb{N}$, that fails when $k \geq 3$. Alternatively, our main result in Chapter 1 uses both a general theorem of Nair and Tenenbaum [24] and the methods of Selberg-Delange [27].

0.1.2 Chapter 2. On the autocorrelation of divisor functions

In Chapter 2, we consider the problem of deriving an analytic condition for the truth of a conjecture of Conrey-Gonek [3], Ng-Thom [25] and Tao [28] on the asymptotic behaviour the sum

$$D_3(x,1) = \sum_{n \leq x} d_3(n) d_3(n+1). \tag{0.1.4}$$

as $x \to \infty$. We address a weaker version of the full conjecture (see Section 2.1.1). Specifically, we consider the statement that (0.1.4) is asymptotic to

$$\prod_p \left(2\left(1-\frac{1}{p}\right)^2 - \left(1-\frac{1}{p}\right)^4 \right) \frac{x \log^4 x}{4} + O\left(x(\log x)^3\right). \tag{0.1.5}$$

The motivation here is that, from an algebraic perspective, this problem is a simpler special case of the “binary additive divisor problem” (as it is called in Ng-Thom [25]), involving the more general sums

$$D_k(x,h) = \sum_{n \leq x} d_k(n) d_k(n+h), \tag{0.1.6}$$

where k is a fixed positive integer and h is bounded by some increasing function of x.

Due to its connection with the problem of the $2k$th moments of the Riemann zeta function (as described by Conrey-Gonek [3] and Ivić [18]) this problem has a long history, although the current status of all these questions are still largely conjectural for $k \geq 3$ and any non-zero value of h.

The main results of Chapter 2 is Theorem 2.1.2. This result is rather too technical to state clearly without further introductory terminology but, roughly speaking, it shows that the conjecture (0.1.5) is equivalent to a specific bound on a particular complex integral of the meromorphic function

$$\psi_3(s,x) = \sum_{q \leq x} \sum_{n \equiv 1(q)} \frac{d_2(q) d_3(n)}{n^s},$$

where $s = \sigma + it$ is the complex variable of integration the path of integration $x = x(T)$ depends on a "height" parameter T.

0.1.3 Chapter 3. On correlation with the Möbius function

In Chapter 3, we consider the problem of deriving an equivalent condition for the *correlation* of an arbitrary bounded *non-trivial*[1] real sequence $\xi(n)$ with the Möbius function $\mu(n)$. Here, correlation is the statement that

$$\sum_{n \leq N} \xi(n) t(n) = \Omega(N),$$

[1] see Definition 3.1.2 for the nomenclature "non-trivial".

where the notation $f(N) = \Omega(g(N))$ indicates that there exist arbitrarily large values of N for which $|f(N)/g(N)| > c$ for some fixed $c > 0$.

The motivation for considering this problem is to address the question of how to investigate the converse of a conjecture of Sarnak [26]. This conjecture predicts that if $\xi(n)$ arises as the sampling sequence of a dynamical system of *zero entropy* then $\xi(n)$ is asymptotically orthogonal to $\mu(n)$. However, quoting an unpublished result of Bourgain, Sarnak [26] states that the converse of this conjecture is false. In light of this, it seems reasonable to seek to find an alternative characterisation of sequences which correlate with $\mu(n)$.

The main result of Chapter 3 is Theorem 3.2.1. Theorem 3.2.1 demonstrates that the correlation of $\xi(mn)$ with $\mu(m)$ for some $n \in \mathbb{N}$ is equivalent to a particular property of the sequence of vectors $\xi = \sum \xi(n)e_n$ in $\mathbb{R}^N$ as $N \to \infty$. By considering explicit examples of $\mathbb{R}^N$, some interesting observations are made.

Chapter 1

On the local maxima of divisor functions

1.1 Introduction

Let n be a natural number and let $d(n)$ denote the number of divisors of n. In their paper [8], Erdős and Hall determined the following asymptotic for the local maxima of $d(n)$:

Theorem 1.1.1. *If* $h = o((\log x)^{3-2\sqrt{2}})$*, then*

$$\sum_{n\leq x} \max\{d(n), d(n+1), ..., d(n+h-1)\} = hx\log x + O(h^2 x(\log x)^{2(\sqrt{2}-1)}). \tag{1.1.1}$$

In the case $h = 2$, (1.1.1) reduces to

$$\sum_{n\le x} \max\{d(n), d(n+1)\} = 2x\log x + O(x(\log x)^{2(\sqrt{2}-1)}). \tag{1.1.2}$$

The leading term on the r.h.s of (1.1.2) was obtained previously by Katai [21], yet with the less precise error term $O(x(\log x)^{1-\delta})$ for some fixed $\delta > 0$. Erdős and Hall derived (1.1.2) by using the identities

$$\max\{x, y\} = x + y - \min\{x, y\} \tag{1.1.3}$$

and $\min\{x, y\} \le \sqrt{xy}$, and by obtaining an upper bound for sums over the geometric means $\sqrt{d(n)d(n+1)}$. Their proof provides us with

$$\sum_{n\le x} \max\{d(n), d(n+h)\} = 2x\log x + O(x(\log x)^{2(\sqrt{2}-1)}), \tag{1.1.4}$$

for fixed values of h, although the authors do not mention this statement explicitly. The lower bound

$$\sum_{n\le x} \min\{d(n), d(n+1)\} = \Omega\left(\frac{x(\log x)^{2(\sqrt{2}-1)}}{(\log\log x)^{44}}\right) \tag{1.1.5}$$

is also proved by Erdős and Hall, and the best known improvement of (1.1.5) is

$$\Omega\left(\frac{x(\log x)^{2(\sqrt{2}-1)}}{\sqrt{\log\log x}}\right), \tag{1.1.6}$$

which is due to Hall and Tenenbaum [13].

Let $k \ge 2$ and consider the error term $E_k(x, h)$ in the asymptotic relation

$$\begin{aligned} \sum_{n\le x} \max\{d_k(n), d_k(n+h)\} &= \sum_{n\le x} d_k(n) + \sum_{n\le x} d_k(n+h) - \sum_{n\le x} \min\{d_k(n), d_k(n+h)\} \\ &= 2\sum_{n\le x} d_k(n) + E_k(x, h) \end{aligned}$$

where

$$\begin{aligned} E_k(x,h) &= -\sum_{n\le x} d_k(n) + \sum_{n\le x} d_k(n+h) - \sum_{n\le x} \min\{d_k(n), d_k(n+h)\} \\ &= -\sum_{n\le x} \min\{d_k(n), d_k(n+h)\} + O_h(x^{\epsilon}). \qquad (1.1.7) \end{aligned}$$

The main result of this chapter is Theorem 1.1.2 below, which is proved in Section 1.3.

Theorem 1.1.2. *If k and h are fixed, then*

$$E_k(x,h) = O(x(\log x)^{2(\sqrt{k}-1)}) \qquad (1.1.8)$$

as $x \to \infty$.

In Section 1.4 we address the following special case of a conjecture of Hall and Tenenbaum [13], and briefly explain how fundamental notions from probabilistic number theory may be relevant to resolving it.

Conjecture 1.1.1 (Hall and Tenenbaum)**.** *As $x \to \infty$ one has*

$$E_2(x,1) = O\left(\frac{x(\log x)^{2(\sqrt{2}-1)}}{\sqrt{\log\log x}}\right). \qquad (1.1.9)$$

Remark 1.1.3. In light of (1.1.6), the truth of Conjecture 1.1.1 implies that the r.h.s of (1.1.9) is the precise order of $E_2(x,1)$.

1.2 The Erdős-Hall method

Erdős and Hall [8] proved that $E_2(x,h) = O(x(\log x)^{2(\sqrt{2}-1)})$ by using a method that relies on a particular growth constraint on the function $\sqrt{d_2(p^\alpha)}$, $\alpha \in \mathbb{N}$, but which fails for $\sqrt{d_k(p^\alpha)}$, $\alpha \in \mathbb{N}$, when $k > 3$. In this section we describe the method used in [8], and why the required growth constraint fails for $k > 3$.

Let $g(n)$ be a non-negative multiplicative function with the property that $g(p^\alpha) \geq g(p^{\alpha-1})$ for $\alpha \geq 1$. Since $\sqrt{g(n)}$ is also multiplicative, we have

$$\sqrt{g(n)} = \sum_{d|n} f(d) \tag{1.2.1}$$

where

$$f(p^\alpha) = \sqrt{g(p^\alpha)} - \sqrt{g(p^{\alpha-1})} \geq 0 \tag{1.2.2}$$

for $\alpha \geq 1$ and $f(1) = 1$.

The method of Erdős and Hall uses the facts that

$$\min\{g(n), g(n+1)\} \leq \sqrt{g(n)g(n+1)} \tag{1.2.3}$$

and

$$\sum_{n\leq x} \sqrt{g(n)g(n+1)} = \sum_{n\leq x} \sum_{d|n} f(d) \sum_{e|n+1} f(e) \tag{1.2.4}$$

then, by obtaining a suitable bound for the r.h.s of (1.2.4), clearly the same bound holds for

$$\sum_{n\leq x} \min\{g(n), g(n+1)\}. \tag{1.2.5}$$

Interchanging the order of summation in (1.2.4) gives

$$\sum_{\substack{d\leq x, e\leq x+1\\(d,e)=1}} f(de) \sum_{\substack{n\leq x/d\\nd\equiv -1 \pmod e}} 1, \tag{1.2.6}$$

where the inner summation on the r.h.s of (1.2.6) is

$$\sum_{\substack{n \le x/d \\ nd \equiv -1 \pmod e}} 1 = \left\lfloor \frac{1}{e} \left\lfloor \frac{x}{d} \right\rfloor \right\rfloor + O(1). \tag{1.2.7}$$

The main term on the r.h.s of (1.2.7) is not dominant when $d \gg x/e$ so, to proceed in this way, it is necessary to employ one of the following strategies: (*a*) use a more accurate expression for the l.h.s of (1.2.7) (for example by using of the theory of Kloosterman sums), or; (*b*) exploit symmetries of $f(n)$ so that the range of summation in (1.2.6) may be replaced by the range $d \ll x/e$.

The strategy of Erdős and Hall [8] follows (*b*) and establishes the existence a constant C such that

$$\sqrt{g(n)} = \sum_{d|n} f(d) \le C \sum_{\substack{d|n \\ d<\sqrt{n}}} f(d) \tag{1.2.8}$$

when $g(n) = d(n)$. To establish (1.2.8), the authors observe that

$$\sum_{\substack{d|n \\ d \ge \sqrt{n}}} f(d) \le \frac{2}{\log n} \sum_{\substack{d|n \\ d \ge \sqrt{n}}} f(d) \log d \le \frac{2}{\log n} \sum_{d|n} f(d) \log d \tag{1.2.9}$$

for any non-negative f, so that it is sufficient to establish the existence of a $C' < 1/2$ such that

$$\sum_{d|n} f(d) \log d \le C' \log n \sum_{d|n} f(d) \tag{1.2.10}$$

because by (1.2.9) we then have

$$\sum_{d|n} f(d) \le \frac{1}{1-2C'} \sum_{\substack{d|n \\ d<\sqrt{n}}} f(d). \tag{1.2.11}$$

Lemma 1.2.1. *There exists a constant $C' < 1/2$ such that*

$$\sum_{d|n} f(d) \log d \leq C' \log n \sum_{d|n} f(d) \tag{1.2.12}$$

if and only if there exists a constant $C'' > 1/2$ such that

$$\sqrt{g(p^\alpha)} \leq \frac{1}{C''\alpha} \sum_{j=0}^{\alpha-1} \sqrt{g(p^j)} \tag{1.2.13}$$

for every p and every $\alpha \geq 1$.

Proof. For any f, by logarithmic differentiation of

$$\sum_{d|n} \frac{f(d)}{d^s} \tag{1.2.14}$$

one finds that

$$\frac{\sum_{d|n} f(d) \log d}{\sum_{d|n} f(d)} = \sum_{p^\alpha || n} \left(\frac{f(p) + 2f(p^2) + \cdots + \alpha f(p^\alpha)}{1 + f(p) + f(p^2) + \cdots + f(p^\alpha)} \right) \log p. \tag{1.2.15}$$

From (1.2.15) it is obvious that the existence of C' in (1.2.12) is equivalent to

$$\sum_{j=0}^{\alpha} j f(p^j) \leq C'\alpha \sum_{j=0}^{\alpha} f(p^j) \tag{1.2.16}$$

for every p and every $\alpha \geq 1$. By (1.2.2) and some elementary analysis, (1.2.16) reduces to (1.2.13). □

Erdős and Hall prove that the growth constraint (1.2.13) holds when $g(n) = d(n)$ so Lemma 1.2.1 applies. This gives a non-trivial estimate of (1.2.4) and implies Theorem 1.1.1. However, there is the following dilemma:

Corollary 1.2.2. *The growth constraint (1.2.13) does not hold for $g(n) = d_k(n)$ when $k > 3$.*

Proof. Since $d_k(p^j) = \binom{j+k-1}{j}$, one need only observe that

$$\sqrt{\binom{7}{4}} \quad > \quad \frac{1}{2}\sum_{j=0}^{3}\sqrt{\binom{3+j}{3}}, \tag{1.2.17}$$

so (1.2.13) fails for $g(n) = d_4(n)$. Similar arguments show that (1.2.13) fails to hold for any $k > 3$. $\square$

1.3 Proof using the theorems of Nair-Tenenbaum and Selberg-Delange

In this section Theorem 1.1.2 is proved. In light of Corollary 1.2.2, it is necessary to find an alternative method to establish a suitable bound for the l.h.s of (1.2.4). This is achieved via Corollary 1.3.2 of Theorem 1.3.1 below, which implies Theorem 1.1.2 because

$$E_k(h,x) \leq \sum_{n\leq x}\sqrt{d_k(n)d_k(n+h)} + \sum_{x<n\leq x+h} d_k(n) + O_h(1) \tag{1.3.1}$$

and because, for fixed h, the sum

$$\begin{aligned}\sum_{x<n\leq x+h} d_k(n) \quad &\ll_{h,k} \quad \max_{n\leq x+h} d_k(n)\\ &\ll_{h,k} \quad k^{C\log(x+h)/\log\log(x+h)}\\ &\ll_{h,k} \quad x^{o_k(1)}\end{aligned} \tag{1.3.2}$$

which is bounded by the first term on the r.h.s of (1.3.1).

Theorem 1.3.1 is a special case of a very general theorem of Nair and Tenenbaum [24] (Theorem 1 therin). Let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity and let A and B be positive constants. Also let $\alpha > 0$ and $\epsilon > 0$ be quantities which may be taken to be arbitrarily small.

Theorem 1.3.1 (Nair-Tenenbaum)**.** *If F_1, F_2 are non-negative arithmetic functions satisfying*

$$F_1(m)F_2(n) \leq \min\{A^{\Omega(mn)}, B(\epsilon)(mn)^{\epsilon}\} \tag{1.3.3}$$

whenever $(m,n) = 1$, then

$$\sum_{x \leq n \leq x+y} F_1(n)F_2(n+h) \ll_{A,B,h,\epsilon} \frac{y}{(\log x)^2} \sum_{mn \leq x} \frac{F_1(m)F_2(n)}{mn} \tag{1.3.4}$$

uniformly for $x^{\alpha} \leq y \leq x$.

To prove Theorem 1.1.2 it will be sufficient to prove the following corollary of Theorem 1.3.1, which amounts to showing that

$$\sum_{mn \leq x} \frac{\sqrt{d_k(m)d_k(n)}}{mn} \ll (\log x)^{2\sqrt{k}}. \tag{1.3.5}$$

Corollary 1.3.2. *For fixed h and k we have*

$$\sum_{n \leq x} \sqrt{d_k(n)d_k(n+h)} = O\left(x(\log x)^{2(\sqrt{k}-1)}\right) \tag{1.3.6}$$

as $x \to \infty$.

Proof. Take $F_1(n) = F_2(n) = \sqrt{d_k(n)}$ in Theorem 1.3.1, so that $F_1(m)F_2(n) = \sqrt{d_k(mn)}$ when $(m,n) = 1$. To begin, we must verify that (1.3.3) holds in this case, i.e. that

$$\sqrt{d_k(n)} \leq \min\{A^{\Omega(n)}, B(\epsilon)n^{\epsilon}\} \tag{1.3.7}$$

when n is squarefree. Since $d_k(p) = k$ it follows that $d_k(n) = k^{\Omega(n)}$, so we have $A = \sqrt{k}$. Since $\Omega(n) = O(\log n/\log\log n)$ as $n \to \infty$ it follows that $k^{\Omega(n)} \leq B(\epsilon)n^{\epsilon}$ for every $\epsilon > 0$, so (1.3.3) holds in this case.

For $\sigma > 1$ let

$$D_k(s) = \sum_{1}^{\infty} \frac{d_k^{1/2}(n)}{n^s}. \tag{1.3.8}$$

By the quantitative version of Perron's formula—a general proof of which is given in Titchmarsh [29] (Lemma 3.12)—one now observes that for $\delta > 0$, $k \geq 2$, $T > 0$ and x not an integer we have

$$\begin{aligned} \sum_{mn \leq x} \frac{F_1(m)F_2(n)}{mn} = \sum_{mn \leq x} \frac{d_k^{1/2}(m)d_k^{1/2}(n)}{mn} &= \frac{1}{2\pi i}\int_{\delta - iT}^{\delta + iT} D_k^2(s+1)\frac{x^s ds}{s} \\ &+ O\left(\frac{x^{\delta}}{T}D_k^2(\delta+1)\right) \\ &+ O\left(\frac{\log x}{T}\max_{n \leq 2x}\frac{1}{n}\sum_{d|n} d_k^{1/2}(d)\right). \end{aligned} \tag{1.3.9}$$

The remaining steps of the proof essentially follow the methods of Selberg [27] and Delange [4], which enable the integral on the r.h.s of (1.3.9) to be estimated. This proceeds by evaluating the integral along segments marginally above and below the potential branch cut $(-\infty, 0]$ and using Hankel's integral representation of $\Gamma(s)$.

The first step is to observe that

$$D_k^2(s) = H_k(s)\zeta^{2k^{1/2}}(s), \tag{1.3.10}$$

where $H_k(s)$ has an absolutely convergent Euler product on compact subsets of the half plane $\sigma > 1/2$. As such, for fixed k, $|H_k(s)|$ is bounded above and away from zero on compact subsets of the half plane $\sigma > 1/2$. Moreover, due to the simple pole of $\zeta(s)$ at $s = 1$, from (1.3.10) it is evident that $(-\infty, 0]$ is a branch cut for $D_k^2(s+1)$ whenever k is not square.

Given $\epsilon > 0$, one takes the path of integration in (1.3.9) to consist of horizontal segments from $\delta - iT$ to $-\delta - iT$ and $-\delta + iT$ to $\delta + iT$, vertical segments from $-\delta - iT$ to $-\delta - i\epsilon$ and $-\delta + i\epsilon$ to $-\delta + iT$, and a truncated Hankel contour (a path from $-\delta - i\epsilon$ to $-\delta + i\epsilon$ passing around the cut along the segment $[-\delta, 0]$, but not crossing it). From (1.3.10), the bounds on $|H_k(s)|$ and the elementary fact that $\zeta(\sigma + it) = O(t^{1-\sigma+\delta})$ for $\sigma \geq 0$, it is immediate that the vertical segments of the integral are

$$\left| \frac{1}{2\pi i} \int_{-\delta+i\epsilon}^{-\delta+iT} \frac{H_k(s+1)\zeta^{2k^{1/2}}(s+1)x^s ds}{s} \right| \ll_{k,\delta} x^{-\delta} T^{4\delta k^{1/2}}, \tag{1.3.11}$$

and that the horizontal segments of the integral are

$$\left| \frac{1}{2\pi i} \int_{-\delta+iT}^{\delta+iT} \frac{H_k(s+1)\zeta^{2k^{1/2}}(s+1)x^s ds}{s} \right| \ll_{k,\delta} x^{\delta} T^{4\delta k^{1/2}-1}. \tag{1.3.12}$$

Taking $T = x^{2\delta}$ and $\delta = k^{-1/2}/8$, the r.h.s. of (1.3.11) is

$$x^{-\delta}(x^{2\delta})^{4\delta k^{1/2}} = x^{-\delta+8\delta^2 k^{1/2}} = x^{-\delta+k^{-1/2}/8} = 1 \tag{1.3.13}$$

and the r.h.s. of (1.3.12) is

$$x^{\delta}(x^{2\delta})^{4\delta k^{1/2}-1} = x^{-\delta+8\delta^2 k^{1/2}} = 1, \tag{1.3.14}$$

so (1.3.11) and (1.3.12) are bounded as $x \to \infty$ for fixed k.

Moreover, with these choices for δ and T, the first error term on the r.h.s of (1.3.9) is

$$\frac{x^{\delta}}{T} D_k^2(\delta+1) = x^{-\delta} D_k^2(\delta+1) \ll_k x^{-\delta} \tag{1.3.15}$$

which is bounded as $x \to \infty$ for fixed k. The second error term on the r.h.s of (1.3.9) is

$$\begin{aligned} \frac{\log x}{T} \max_{n \le 2x} \frac{1}{n} \sum_{d|n} d_k^{1/2}(d) \quad &\ll_k \quad x^{-2\delta} \log x (k+1)^{C \log x / \log\log x} \\ &\ll_k \quad x^{-2\delta + C \log k / \log\log x}, \end{aligned} \tag{1.3.16}$$

which is also bounded as $x \to \infty$ for fixed k.

For fixed k then, it follows that

$$\sum_{mn \le x} \frac{d_k^{1/2}(m) d_k^{1/2}(n)}{mn} = \frac{1}{2\pi i} \int_{\mathcal{H}(k,\epsilon)} D_k^2(s+1) \frac{x^s ds}{s} + O_k(1), \tag{1.3.17}$$

where the path of integration $\mathcal{H}(k,\epsilon)$ is from $-k^{-1/2}/8 - i\epsilon$ to $-k^{-1/2}/8 + i\epsilon$ and not intersecting the half line $(-\infty, 0]$. Invoking (1.3.10) and the fact that $\zeta(s)$ has a simple pole at $s = 1$, one may expand $H_k(s+1)$ in a power series about $s = 0$ to give

$$D_k^2(s+1) = \sum_{n \le 2k^{1/2}} c_n s^{n-2k^{1/2}} + O_k(1) \tag{1.3.18}$$

so the r.h.s of (1.3.17) is

$$\sum_{n\le 2k^{1/2}}\frac{c_n}{2\pi i}\int_{\mathcal{H}(k,\epsilon)}x^s s^{n-2k^{1/2}-1}ds+O_k(1). \tag{1.3.19}$$

Making the change of variable $s=z/\log x$ in (1.3.19) then gives

$$\sum_{n\le 2k^{1/2}}\frac{c_n(\log x)^{2k^{1/2}-n}}{2\pi i}\int_{\mathcal{H}(k,\epsilon,x)}e^z z^{n-2k^{1/2}-1}dz+O_k(1), \tag{1.3.20}$$

where $\mathcal{H}(k,\epsilon,x)$ indicates a path of integration from $-k^{-1/2}\log x/8-i\epsilon\log x$ to $-k^{-1/2}\log x/8+i\epsilon\log x$ and not intersecting the half line $(-\infty,0]$. Taking $\epsilon=o(1/\log x)$, the path $\mathcal{H}(k,\epsilon,x)$ approaches a standard Hankel contour $\mathcal{H}$ as $x\to\infty$ therefore, using Hankel's identity

$$\frac{1}{\Gamma(s+1)}=\frac{1}{2\pi i}\int_{\mathcal{H}}e^z z^{-s-1}dz, \tag{1.3.21}$$

in (1.3.20), from (1.3.9) we now have

$$\begin{aligned}\sum_{mn\le x}\frac{d_k^{1/2}(m)d_k^{1/2}(n)}{mn} &= \sum_{n\le 2k^{1/2}}\frac{c_n(\log x)^{2k^{1/2}-n}}{\Gamma(2k^{1/2}-n+1)}+O_k(1)\\ &= O_k\left((\log x)^{2k^{1/2}}\right).\end{aligned} \tag{1.3.22}$$

Thus, (1.3.22) and (1.3.4) together give

$$\sum_{x\le n\le x+y}d_k^{1/2}(n)d_k^{1/2}(n+h)\ll_{h,k}y(\log x)^{2(k^{1/2}-1)} \tag{1.3.23}$$

uniformly for $x^\alpha\le y\le x$.

To complete the proof of Corollary 1.3.2 we take $y=x=2^{-m-1}X$ successively in (1.3.23) and sum over the range $0\le m\le\log_2 X$, which gives

$$\begin{aligned}\frac{\sum_{n\le X}d_k^{1/2}(n)d_k^{1/2}(n+h)}{X(\log X)^{2(k^{1/2}-1)}} &\ll_{h,k} \sum_{0\le m\le\log_2 X}2^{-m-1}\left(1-\frac{(m-1)\log 2}{\log X}\right)^{2(k^{1/2}-1)}\\ &\ll_k 1\end{aligned} \tag{1.3.24}$$

as $X \to \infty$. $\square$

1.4 On a conjecture of Hall-Tenenbaum

We conclude this chapter by addressing a special case of a conjecture of Hall-Tenenbaum [13], which is equivalent to the statement that

$$E_2(x, 1) \asymp \frac{x(\log x)^{2(\sqrt{2}-1)}}{\sqrt{\log\log x}}. \tag{1.4.1}$$

In particular, we briefly indicate how elementary notions from probabilistic number theory may be applicable to this conjecture.

We begin by considering a general statement. Let $f(n)$ be a bounded real sequence and let $g(n)$ be a positive real sequence such that

$$\lim_{x\to\infty} \sum_{n\le x} g(n) = \infty. \tag{1.4.2}$$

Consider the quotient

$$Q(x) = \lim_{x\to\infty} \frac{\sum_{n\le x} f(n)g(n)}{\sum_{n\le x} g(n)}, \tag{1.4.3}$$

for which we can prove the following "abelian" lemma:

Lemma 1.4.1. *If $f(n) \to l$ converges, then $Q(x) \to l$.*

Proof. Write

$$\sum_{n\le x} f(n)g(n) = f(x)\sum_{n\le x} g(n) + \sum_{n\le x}(f(n) - f(x))g(n) \tag{1.4.4}$$

then, since $f(x) = l + o(1)$ as $x \to \infty$, one has

$$\begin{aligned}\sum_{n\le x} f(n)g(n) &= l\sum_{n\le x} g(n) + \sum_{n\le N}(f(n)-f(x))g(n) + \sum_{N<n\le x}(f(n)-f(x))g(n)\\ &+ o\left(\sum_{n\le x} g(n)\right) \end{aligned} \tag{1.4.5}$$

for fixed N. We have

$$\sum_{n\le N}(f(n)-f(x))g(n) = O\left(\sum_{n\le N} g(n)\right) = o\left(\sum_{n\le x} g(n)\right) \tag{1.4.6}$$

by boundedness of $f(n)$ and (1.4.2). Finally, since $f(n)$ converges, for every $\epsilon > 0$ we may choose $N = N(\epsilon)$ such that $|f(n) - f(x)| < \epsilon$ for all $n, x > N$, so

$$\sum_{N<n\le x}(f(n)-f(x))g(n) = o\left(\sum_{n\le x} g(n)\right) \tag{1.4.7}$$

and thus

$$\sum_{n\le x} f(n)g(n) = (l + o(1))\sum_{n\le x} g(n). \tag{1.4.8}$$

□

Now, weakening the assumption of convergence in Lemma 1.4.1 and instead assuming that

$$\frac{1}{x}\sum_{n\le x} f(n) \asymp M_f(x) \tag{1.4.9}$$

for some monotonic function $M_f(x)$, it is reasonable to expect that a statement analogous to Lemma 1.4.1 holds, i.e.

$$\frac{\sum_{n\le x} f(n)g(n)}{\sum_{n\le x} g(n)} \asymp M_f(x) \tag{1.4.10}$$

under suitable conditions on the sequence $g(n)$. In the case of the conjecture of Hall-Tenenbaum (1.4.1), specifically we take

$$f(n) = e^{-|\log(d(n)/d(n+1))|/2}$$

and $g(n) = \sqrt{d(n)d(n+1)}$ so that $f(n)g(n) = \min\{d(n), d(n+1)\}$ and

$$E_2(x,1) = \sum_{n\le x} f(n)g(n). \tag{1.4.11}$$

If it can be shown that (1.4.9) and (1.4.10) hold for this choice of $f(n)$ and $g(n)$, then the problem of proving (1.4.1) is reduced to proving the following two asymptotics:

$$\sum_{n\le x} e^{-|\log(d(n)/d(n+1))|/2} \asymp \frac{x}{(\log\log x)^{1/2}} \tag{1.4.12}$$

and

$$\sum_{n\le x} d^{1/2}(n)d^{1/2}(n+1) \asymp x(\log x)^{2(\sqrt{2}-1)}. \tag{1.4.13}$$

The upper bound implied by (1.4.13) is certainly true (e.g Theorem 1.1.2), and it is likely that the implied lower bound could be proved in a similar way to the theorem of Ingham [17]. Moreover, noting that

$$e^{-|\log(d(n)/d(n+1))|/2} = \min\left\{\sqrt{\frac{d(n)}{d(n+1)}}, \sqrt{\frac{d(n+1)}{d(n)}}\right\}, \tag{1.4.14}$$

it is apparent that (1.4.12) is analogous to the statement that on average

$$\min\left\{\frac{d(n)}{d(n+1)}, \frac{d(n+1)}{d(n)}\right\} \sim \frac{1}{\log\log n}. \tag{1.4.15}$$

It seems plausible that (1.4.15) could be investigated in probabilistic terms, via the theorems of Hardy-Ramanujan [12] or Erdős-Kac [7], although I have not yet attempted to do this. However, I have carried out calculations for all $n \leq 10^9$ which appear to verify (1.4.15) in this range. Of course we must be very skeptical about such numerical "evidence" because the function $\log \log n$ barely increases in this range.

Chapter 2

On the autocorrelation of divisor functions

2.1 Introduction

In this chapter we consider the problem of deriving an analytical condition for the conjectured leading term (as conjectured by Conrey-Gonek [3], Ng-Thom [25], Tao [28]) for the asymptotic behaviour the sum

$$\sum_{n \leq x} d_3(n) d_3(n+1). \tag{2.1.1}$$

This conjecture is a special case of general conjecture of Conrey-Gonek [3] (see Section 2.1.1 herein for the more general form of the conjecture), and states that

Conjecture 2.1.1. *As $x \to \infty$, (2.1.1) is asymptotic to*

$$\prod_p \left(2\left(1 - \frac{1}{p}\right)^2 - \left(1 - \frac{1}{p}\right)^4 \right) \frac{x \log^4 x}{4} + O\left(x(\log x)^3\right). \tag{2.1.2}$$

The motivation for considering this problem is two-fold. Firstly, no conditions of this type have yet appeared in the literature. The second motivation is that, from an algebraic perspective, Conjecture 2.1.1 is relatively simple in comparison with the general form, which involves the more general sums

$$D_k(x,h) = \sum_{n \le x} d_k(n) d_k(n+h), \tag{2.1.3}$$

where k is a fixed positive integer and $h > 0$ is allowed to depend on x. Indeed, the case $h = 1$ is unique in the sense that n and $n + h$ is co-prime for all n, so that the number of solutions to the equations to be solved are independent of h. Moreover, due to the multiplicative nature of $d_k(n)$, these are the only cases in which $d_k(n)d_k(n+h) = d_k(n(n+h))$ for all n.

2.1.1 The binary additive divisor problem

The more general problem of determining the asymptotic behaviour of

$$D_k(x,h) = \sum_{n \le x} d_k(n) d_k(n+h) \tag{2.1.4}$$

is known as the “binary additive divisor problem” (Ng-Thom [25]). Due to it’s connection with the problem of the $2k$th moments of the Riemann zeta function (as described by Conrey-Gonek [3] and Ivić [18]), the binary additive divisor problem is an important and difficult problem which has a long history. A proper survey of the history of this problem is a major undertaking, and will not be given here. For detailed information the reader is referred to the articles of Ivić [19] and Ng-Thom [25].

For $k = 2$, the problem was solved by the work of Ingham [17] and Estermann [6], leading to an asymptotic formula

$$D_2(x, h) = xP_2(\log x, h) + O\left(x^{11/12+\epsilon}\right) \tag{2.1.5}$$

where $P_2(u, h)$ is an explicit polynomial in u of degree 2 depending on h in a specific way. Heath-Brown [14] further improved the error term in (2.1.5) to $O\left(x^{5/6}\right)$ and Motohashi [23] has since improved this to $O(x^{2/3+\epsilon})$ uniformly for $h \leq x^{20/27}$.

For fixed $k \geq 3$, a special case of the general theorem of Nair-Tenenbaum (Theorem 1.3.1 here) implies that

$$D_k(x, h) = O\left(x(\log x)^{2k-2}\right) \tag{2.1.6}$$

(with uniformity aspects in h following from the work of Henriot [15]). A central conjecture (due to Conrey-Gonek [3]) is that

$$D_k(x, h) = xP_{2k-2}(\log x, h) + O\left(x^{1/2+\epsilon}\right) \tag{2.1.7}$$

where $P_m(u, v)$ is polynomial in u of degree m. This is obtained using the circle method, under the hypothesis that the contribution of minor arcs is relatively small. Based on probabilistic reasoning involving psuedorandomness heuristics for primes, Tao [28] conjectured that the leading coefficient is the function

$$c(h,k) = \frac{f_k(h)}{(k-1)!^2} \prod_p \left(2\left(1-\frac{1}{p}\right)^{k-1} - \left(1-\frac{1}{p}\right)^{2k-2} \right), \tag{2.1.8}$$

where $f_k(h)$ is an explicit multiplicative function of h. Moreover, Ng-Thom [25] obtained the same prediction from a rather different probabilistic perspective, yet the current status of the general problem is still largely conjectural for $k \geq 3$ and any non-zero value of h.

We conclude this section by noting that Andrade, Soroker and Rudnick [1] have established an analogous result in the function field setting (Theorem 1.1 therein).

2.1.2 Main theorems

Let $s = \sigma + it$ and let X be a positive number.

Definition 2.1.1. *For $k \in \mathbb{N}$ and $\sigma > 1$ let*

$$\psi_k(s,X) = \sum_{1}^{\infty} \frac{d_k(n)}{n^s} \sum_{\substack{q \leq X \\ q \mid n-1}} d_{k-1}(q) - \sum_{q \leq X} d(q). \tag{2.1.9}$$

For $s \in \mathbb{C}/\{1\}$, we note that we also have also have the representation

$$\psi_k(s,X) = \sum_{q \leq X} \frac{d_{k-1}(q)}{\phi(q)} \sum_{\chi \ (mod\ q)} L^k(s,\chi) - \sum_{q \leq X} d(q), \tag{2.1.10}$$

which is established in (2.2.7). The main result of this chapter is Theorem 2.1.2, which shows that Conjecture 2.1.1 holds if and only if a specific upper bound on an integral of $\psi_3(s,X)$ holds, where X depends on the range of integration:

Theorem 2.1.2. *Conjecture 2.1.1 holds if and only if*

$$\int_{\bar{z}(x)}^{z(x)} \frac{\psi_3(s,x)x^s ds}{s} = O\left(x(\log x)^3\right), \tag{2.1.11}$$

where $z(x) = 1 + (\log x)^{-3/5} + ie^{(\log x)^{2/5}}$ *and the path of integration intersects the real axis only in the interval* $(0,1)$.

2.2 Lemmas. Proofs of the main theorems

We begin this section by stating and proving the auxiliary Lemmas (Lemmas 2.2.1 - 2.2.5). The proof of Theorem 2.1.2, which relies on Lemmas 2.2.1 - 2.2.5, is given afterwards.

Lemma 2.2.1. *For* $\sigma > 1$ *let*

$$\psi_k(s) = \lim_{X\to\infty} \psi_k(s,X) = \sum_{2}^{\infty} \frac{d_k(n)d_k(n-1)}{n^s}. \tag{2.2.1}$$

For $c > 1$, $k \geq 2$, $T > 0$ *and* x *not an integer, we have*

$$\begin{aligned}\sum_{1<n\leq x} d_k(n)d_k(n-1) &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\psi_k(s)x^s ds}{s} \\ &+ O\left(\frac{x^c}{T(c-1)^{2k-1}}\right) + O\left(\frac{x\log x}{T}\max_{n\leq 2x} d_k(n)d_k(n-1)\right).\end{aligned} \tag{2.2.2}$$

Proof. Equation (2.2.2) is a quantitative version of Perron's formula, a general proof of which is given in Titchmarsh [29] (Lemma 3.12 therein). The exponent $2k-1$ of $c-1$ in the denominator of the first error term on the r.h.s of (2.2.2) appears because, firstly; the point $s = 1$ is a singularity of $\psi_k(s)$ by a general theorem of Landau [22] on Dirichlet series with all but finitely many coefficients being positive and, secondly; it is a well-known consequence of Theorem 1.3.1 that this singularity is a pole of order at most $2k - 1$. □

Lemma 2.2.2. *We have*

$$\begin{aligned}\sum_{1<n\leq x} d_k(n)d_k(n-1) &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\psi_k(s,x)x^s ds}{s} \\ &+ O\left(\frac{x^c}{T(c-1)^{2k-1}}\right) + O\left(\frac{x\log x}{T}\max_{n\leq 2x} d_k(n)d_k(n-1)\right).\end{aligned} \tag{2.2.3}$$

Proof. The contribution on the l.h.s of (2.2.2) remains the same when $\psi_k(s)$ is replaced with $\psi_k(s,x)$. The error terms are also the same as in (2.2.2) because if $c > 1$ we have

$$\begin{aligned}\psi_k(c) - \psi_k(c,x) &= \sum_1^\infty \frac{d_k(n)}{n^c} \sum_{\substack{q>x \\ q|n-1}} d_{k-1}(q) \\ &\leq \sum_1^\infty \frac{d_k(n)}{n^c} \sum_{q|n-1} d_{k-1}(q) \\ &= \psi_k(c).\end{aligned} \tag{2.2.4}$$

□

Lemma 2.2.3. *For $k \geq 2$ we have*

$$\begin{aligned}\sum_{1<n\leq x} d_k(n)d_k(n-1) &= \sum_{q\leq x}\frac{d_{k-1}(q)}{\phi(q)}\mathrm{res}_{s=1}\left(\frac{L^k(s,\chi_0)x^s}{s}\right) + \frac{1}{2\pi i}\int_{\bar{z}_k(x)}^{z_k(x)} \frac{\psi_k(s,x)x^s ds}{s} \\ &+ O\left(x(\log x)^{2k-3}\right).\end{aligned} \tag{2.2.5}$$

where

$$z_k(x) = 1 + (\log x)^{-(2k-3)/(2k-1)} + ie^{(\log x)^{2/(2k-1)}} \tag{2.2.6}$$

and the path of integration intersects the real axis only in $(0, 1)$.

Proof. By Cauchy's theorem we move the path of integration between $c - iT$ and $c + iT$ in (2.2.3), intersecting the real axis in the interval $(0, 1)$. To evaluate the residue at $s = 1$, for $\sigma > 1$ we have

$$\begin{aligned}
\psi_k(s, x) &= \sum_1^\infty \frac{d_k(n)}{n^s} \sum_{\substack{q \le x \\ q | n-1}} d_{k-1}(q) - \sum_{q \le x} d(q) \\
&= \sum_{q \le x} d_{k-1}(q) \sum_{n \equiv 1 \pmod q} \frac{d_k(n)}{n^s} - \sum_{q \le x} d(q) \\
&= \sum_{q \le x} \frac{d_{k-1}(q)}{\phi(q)} \sum_{\chi \pmod q} \sum_1^\infty \frac{\chi(n) d_k(n)}{n^s} - \sum_{q \le x} d(q) \\
&= \sum_{q \le x} \frac{d_{k-1}(q)}{\phi(q)} \sum_{\chi \pmod q} L^k(s, \chi) - \sum_{q \le x} d(q) \qquad (2.2.7)
\end{aligned}$$

so, since $L(s, \chi)$ extends to an analytic function in a neighbourhood of $s = 1$ (except for those L-functions for the principal characters which have a simple pole at $s = 1$), the sum on the r.h.s of (2.2.7) defines $\psi_k(s, x)$ for all $s \in \mathbb{C}/\{1\}$. As such, the residue of $\psi_k(s, x)x^s/s$ at $s = 1$ is equal to the sum of the residues of the terms corresponding to the principal character for each q.

To bound the error term on the r.h.s of (2.2.3), we take $c = 1+(\log x)^{-(2k-3)/(2k-1)}$ and $\log T = (\log x)^{2/(2k-1)}$. This gives

$$\frac{x^c}{T(c-1)^{2k-1}} = x(\log x)^{2k-3} \tag{2.2.8}$$

and

$$\frac{x \log x}{T} = x^{2-c} \log x = xe^{-(\log x)^{2/(2k-1)}} \log x = o(x(\log x)^{-A}) \tag{2.2.9}$$

for any $A < \infty$. $\square$

We now specialise to the case $k = 3$ to evaluate the sum on the r.h.s of (2.2.5).

Lemma 2.2.4. *We hace*

$$\sum_{q \le x} \frac{d(q)}{\phi(q)} \mathrm{res}_{s=1} \left(\frac{L^3(s, \chi_0) x^s}{s} \right) = \frac{1}{2} x (\log x)^2 \sum_{q \le x} \frac{d(q)\phi^2(q)}{q^3} + O(x(\log x)^3) \tag{2.2.10}$$

Proof. Let

$$\varphi_s(q) = \prod_{p|q} \left(1 - \frac{1}{p^s}\right), \tag{2.2.11}$$

so that $L(s, \chi_0) = \varphi_s(q)\zeta(s)$, and denote by $\varphi^{(j)}(q)$ the derivative of $\varphi_s(q)$ at $s = 1$. Thus, by Lemma 2.2.3 we have

$$\begin{aligned}
\mathrm{res} \left(\frac{\varphi_s^3(q)\zeta^3(s)x^s}{s} \right) &= \frac{1}{2} \lim_{s \to 1} \frac{d^2}{ds^2} \frac{((s-1)\zeta(s))^3 \varphi_s^3(q) x^s}{s} \\
&= \frac{1}{2} x (\log x)^2 \varphi^3(q) \\
&+ x \log x \left(3\varphi^2(q)\varphi^{(1)}(q) + (3\gamma - 1)\varphi^3(q) \right) + O\left(x \log X\right)
\end{aligned} \tag{2.2.12}$$

uniformly for $q \le x$. Therefore, the l.h.s of (2.2.10) is

$$\begin{aligned}
&= \frac{1}{2} x (\log x)^2 \sum_{q \le x} \frac{d(q)\varphi^3(q)}{\phi(q)} + 3x \log x \sum_{q \le x} \frac{d(q)\varphi^2(q)\varphi(q)^{(1)}}{\phi(q)} \\
&+ (3\gamma - 1) x \log x \sum_{q \le x} \frac{d(q)\varphi^3(q)}{\phi(q)} + O\left(x \log x \sum_{q \le x} \frac{d(q)}{\phi(q)} \right).
\end{aligned} \tag{2.2.13}$$

Denoting Euler's totient function by $\phi(q)$, using the fact that $\varphi(q) = \phi(q)/q$ gives

$$\begin{aligned} &= \frac{1}{2}x(\log x)^2 \sum_{q \le X} \frac{d(q)\phi^2(q)}{q^3} + 3x\log x \sum_{q\le x} \frac{d(q)\phi(q)\varphi(q)^{(1)}}{q^2} \\ &+ (3\gamma - 1)x\log x \sum_{q\le x} \frac{d(q)\phi^2(q)}{q^3} + O\left(x\log x \sum_{q\le x} \frac{d(q)}{\phi(q)} \right). \end{aligned} \tag{2.2.14}$$

Since $\phi(q) \le q$, the third term on the r.h.s of (2.2.14) is $O(x(\log x)^3)$. The second term can also be estimated as follows:

$$\begin{aligned} \left| \sum_{q\le x} \frac{d(q)\phi(q)\varphi(q)^{(1)}}{q^2} \right| &\le \sum_{q\le x} \frac{d(q)\phi(q)}{q^2} \sum_{d|q} \frac{\log d}{d} \le \sum_{q\le x} \frac{d(q)}{q} \sum_{d|q} \frac{\log d}{d} \\ &= O\left(\sum_{q\le x} \frac{\log q}{q} \sum_{d|q} \frac{\log d}{d} \right) = O\left(\sum_{d\le x} \frac{\log d}{d^2} \sum_{q\le x/d} \frac{\log(dq)}{q} \right) \\ &= O\left(\log x \sum_{d\le x} \frac{\log d}{d^2} \sum_{q\le x/d} \frac{1}{q} \right) = O\left((\log x)^2 \right). \end{aligned} \tag{2.2.15}$$

Thus, (2.2.14) is

$$\frac{1}{2}x(\log x)^2 \sum_{q\le x} \frac{d(q)\phi^2(q)}{q^3} + O\left(x\log x \sum_{q\le x} \frac{d(q)}{\phi(q)} \right) + O\left(x(\log x)^3 \right). \tag{2.2.16}$$

The sum in the first error term on the r.h.s of (2.2.16) can also be bounded using the identity

$$\frac{q}{\phi(q)} = \sum_{d|q} \frac{\mu^2(d)}{\phi(d)}, \tag{2.2.17}$$

which is obtained by elementary manipulation of the definition of $\phi(q)$. We proceed as follows:

$$
\begin{aligned}
\sum_{q\le x}\frac{d(q)}{\phi(q)} &= \sum_{q\le x}\frac{d(q)}{q}\sum_{d|q}\frac{\mu^2(d)}{\phi(d)} = \sum_{d\le x}\frac{\mu^2(d)}{\phi(d)d}\sum_{q\le x/d}\frac{d(dq)}{q} \\
&= O\left(\log x\sum_{d\le x}\frac{\mu^2(d)}{\phi(d)d}\sum_{q\le x/d}\frac{1}{q}\right) \\
&= O\left(\log x\sum_{d\le x}\frac{\mu^2(d)}{\phi(d)d}\sum_{q\le x}\frac{1}{q}\right) \\
&= O\left((\log x)^2\right)
\end{aligned}
\tag{2.2.18}
$$

by convergence of the sum over d as $x \to \infty$. This completes the proof. □

Lemma 2.2.5. *One has*

$$
\sum_{q\le x}\frac{d(q)\phi^2(q)}{q^3} = \prod_p\left(2\left(1-\frac{1}{p}\right)^2-\left(1-\frac{1}{p}\right)^4\right)\frac{(\log x)^2}{2} + O\left(\log X\right). \tag{2.2.19}
$$

Proof. To prove this, let $\mu_2(n)$ denote the Dirichlet coefficient of $\zeta^{-2}(s)$, so that

$$
\sum_{m|n}\mu_2(m)d\left(\frac{n}{m}\right) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise.}\end{cases} \tag{2.2.20}
$$

Thus

$$
\begin{aligned}
\sum_{q\le x}\frac{d(q)\phi^2(q)}{q^3} &= \sum_{q\le x}\frac{1}{q}\sum_{n|q}\frac{d(n)\phi^2(n)}{n^2}\sum_{m|\frac{q}{n}}\mu_2(m)d\left(\frac{q}{mn}\right) \\
&= \sum_{q\le x}\frac{1}{q}\sum_{m|q}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right)d\left(\frac{q}{m}\right) \\
&= \sum_{m\le x}\frac{1}{m}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right)\sum_{q\le x/m}\frac{d\left(q\right)}{q}.
\end{aligned}
\tag{2.2.21}
$$

Now, it is elementary that (e.g. Dirichlet)

$$
\sum_{q\le x}\frac{d\left(q\right)}{q} = \frac{(\log x)^2}{2} + C\log x + D + O(x^{-\delta}) \tag{2.2.22}
$$

where C, D and δ are positive constants. Thus (2.2.21) is

$$\begin{aligned}
&= \frac{1}{2}\sum_{m\le x}\frac{1}{m}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right)\left(\log\left(\frac{x}{m}\right)\right)^2 \\
&+ C\sum_{m\le x}\frac{1}{m}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right)\log\left(\frac{x}{m}\right) \\
&+ D\sum_{m\le x}\frac{1}{m}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right)+O(1). \qquad (2.2.23)
\end{aligned}$$

Now let r be real and consider the formal Dirichlet series

$$\begin{aligned}
&\sum_1^\infty \frac{1}{m^{r+1}}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right) \\
&= \sum_1^\infty \frac{\mu(m)}{m^{r+1}}\sum_1^\infty \frac{d(n)\phi^2(n)}{n^{r+3}} \\
&= \prod_p\left(\left(1-\frac{1}{p^{r+1}}\right)^2+\left(1-\frac{1}{p}\right)^2\left(1-\left(1-\frac{1}{p^{r+1}}\right)^2\right)\right) \\
&= \prod_p\left(1-\frac{4}{p^{r+2}}+\frac{2}{p^{r+3}}+\frac{2}{p^{2r+3}}-\frac{1}{p^{2r+4}}\right). \qquad (2.2.24)
\end{aligned}$$

We may observe that the exponents of $1/p$ in the factors of the Euler product on the r.h.s of (2.2.24) are all greater than 1 whenever $r > -1$ so this Euler product and the associated Dirichlet series converge absolutely for $r > -1$, and therefore also uniformly on compact subsets of the open half plane $\Re r > -1$ for complex values of r. Thus, since all the derivatives

$$(-1)^j\sum_1^\infty \frac{(\log m)^j}{m^r}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right) \qquad (2.2.25)$$

exist in a neighbourhood of $r = 0$, we conclude that (2.2.23) is

$$\frac{(\log x)^2}{2}\sum_{m\le x}\frac{1}{m}\left(\sum_{n|m}\mu_2\left(\frac{m}{n}\right)\frac{d(n)\phi^2(n)}{n^2}\right)+O(\log x). \qquad (2.2.26)$$

Setting $r = 0$ in (2.2.24), we find that

$$\sum_{1}^{\infty} \frac{1}{m} \left(\sum_{n|m} \mu_2 \left(\frac{m}{n} \right) \frac{d(n)\phi^2(n)}{n^2} \right) = \prod_p \left(2\left(1 - \frac{1}{p}\right)^2 - \left(1 - \frac{1}{p}\right)^4 \right) \tag{2.2.27}$$

and (2.2.19) follows. □

Proof of Theorem 2.1.2. By Lemma 2.2.2, Cauchy's theorem implies that the integral on the r.h.s of (2.2.3) may be expressed in terms of the residue of $\psi_k(s, x)x^s/s$ at $s = 1$. We may choose $c = c(x)$ and $T = T(x)$ as described in Lemma 2.2.3, so that the error terms on the r.h.s of (2.2.3) are $O(x(\log x)^{2k-3})$. Setting $k = 3$ in lemmas 2.2.3–2.2.5 completes the proof of Theorem 2.1.2. □

2.3 Extension to all values of k

We conclude this chapter by observing that the auxiliary Lemmas 2.2.1 - 2.2.3 are valid for all values of k, so that generalising Theorem 2.1.2 depends on generalising Lemma 2.2.4 and Lemma 2.2.5 to all values of k. Since it appears to be possible to carry out the computations for each k, there appears to be no reason why similar conclusions would not hold. Thus, we conjecture that a similar equivalent condition for the truth of the general conjecture (2.1.2) of Conrey and Gonek also holds for all values of k, that is:

Conjecture 2.3.1.

$$\begin{aligned} D_k(x, -1) &= \prod_p \left(2\left(1 - \frac{1}{p}\right)^{k-1} - \left(1 - \frac{1}{p}\right)^{2k-2} \right) \frac{x(\log x)^{2k-2}}{(k-1)!^2} \\ &+ O\left(x(\log x)^{2k-3}\right) \end{aligned} \tag{2.3.1}$$

if and only if

$$\int_{\bar{z}_k(x)}^{z_k(x)} \frac{\psi_k(s,x)x^s ds}{s} = O(x(\log x)^{2k-3}), \tag{2.3.2}$$

where

$$z_k(x) = 1 + (\log x)^{-(2k-3)/(2k-1)} + ie^{(\log x)^{2/(2k-1)}} \tag{2.3.3}$$

and the path of integration intersects the real axis in only in the interval $(0, 1)$*.*

Chapter 3

On correlation with the Möbius function

3.1 Introduction

We denote by $\omega(n)$ the number of distinct prime factors of the natural number n and define the *Möbius function* $\mu(n)$ by

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is squarefree} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.1}$$

In this chapter we consider the problem of deriving a condition for an arbitrary bounded *non-trivial* real sequence $\xi(n)$ to *correlate* with $\mu(n)$, where *correlation* is defined in Definitions 3.1.1 and 3.1.2 below. The notation $f(N) = \Omega(g(N))$ indicates that there exist arbitrarily large values of N for which $|f(N)/g(N)| > c$

for some fixed $c > 0$. We shall say that

Definition 3.1.1. *$\xi(n)$ correlates with $\mu(n)$ if*

$$\sum_{n \leq N} \mu(n)\xi(n) = \Omega(N). \tag{3.1.2}$$

Otherwise, $\xi(n)$ and $\mu(n)$ are said to be asymptotically orthogonal.

Definition 3.1.2. *$\xi(n)$ is non-trivial if*

$$\sum_{n \leq N} |\xi(n)|^2 = \Omega(N). \tag{3.1.3}$$

The main result of this chapter is Theorem 3.2.1. Specifically, rather than working with Definition 3.1.1 directly, Theorem 3.2.1 addresses the question: *when does $\xi(mn)$ correlate with $\mu(m)$ for some natural number n*? The reason for considering this variant of the problem is that a non-multiplicative sequence $\xi(n)$ may exhibit similar complexity as $\mu(n)$ for those values of n for which $\mu(n) = 0$, while remaining asymptotically orthogonal to $\mu(n)$ in the sense of Definition 3.1.1.

Example 3.1.3*.* As a simple example, if one takes

$$\xi(n) = \begin{cases} \mu(k) & n = 4k \\ 1 & \text{otherwise} \end{cases} \tag{3.1.4}$$

then $\xi(n)$ is asymptotically orthogonal to $\mu(n)$.

To avoid artificial examples such as (3.1.4), we work with the stronger requirement of having $\xi(mn)$ asymptotically orthogonal to $\mu(m)$ *for all natural numbers* n.

Before stating the results, we shall introduce some preliminary notions and notation. We begin by describing the *Möbius randomness principle* and the motivation for considering this problem.

3.1.1 The Möbius randomness principle. Motivation

A central reason for studying sums of the form (3.1.2) is that, owing to identities such as

$$\Lambda(n) = \left\{ \begin{array}{ll} \log p & n = p^j \\ 0 & \text{otherwise} \end{array} \right\} = -\sum_{d|n} \mu(d) \log d \tag{3.1.5}$$

where $\Lambda(n)$ is the *Von-Mangoldt function*, the Möbius function $\mu(n)$ is linked with fundamental questions about primes. Such questions can often be expressed equivalently as questions about sums involving $\mu(n)$. For instance, de la Vallée Poussin [30] showed that there is a $c > 0$ such that

$$\sum_{n \leq N} \mu(n) = o\left(Ne^{-c\sqrt{\log N}}\right), \tag{3.1.6}$$

which implies a non-trivial zero-free region for the Riemann zeta function and the Prime Number Theorem. Also Davenport [10] showed that

$$\sum_{n \leq N} \mu(n) e^{in\theta} = o\left(\frac{N}{\log^c N}\right) \tag{3.1.7}$$

for any given $c < \infty$ and uniformly in θ, which implies the corresponding zero-free regions for Dirichlet L-functions and the prime number theorem in arithmetic progressions.

The modern viewpoint is that information about the complexity of $\mu(n)$ may be inferred indirectly from the asymptotic behaviour of (3.1.1) when the complexity of $\xi(n)$ is known.

Definition 3.1.4 (Low complexity). *A sequence $\xi(n)$ is said to be of low complexity if $\xi(n)$ is asymptotically orthogonal to $\mu(n)$.*

Example 3.1.5. Since

$$\sum_{n \le N} \mu^2(n) \sim 6N/\pi^2,$$

the Möbius function itself is not of low complexity.

Definition 3.1.4 is motivated by a meta-principle—known as the *Möbius randomness principle*—that a broad class of non-trivial bounded functions are insufficiently complex to interfere with the complexity of the sign changes of $\mu(n)$ (in other words, insufficiently complex to correlate with $\mu(n)$).

Based on the work of Furstenberg [11], P. Sarnak [26] has formulated a fruitful conjecture. Essentially, this is the notion that identifying a sequence $\xi(n)$ as the *sampling sequence* of a dynamical system allows one to identify its complexity with the *topological entropy* of the dynamical system.

Definition 3.1.6. *A flow F is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a continuous map.*

Definition 3.1.7. *The sampling sequences associated with a flow F are the sequences $\xi(n) = f(T^n x)$ for some $x \in X$ and $f \in C(X)$.*

Definition 3.1.8. *The topological entropy of a flow is the quantity*

$$h(X,T) = \lim_{\epsilon \to 0} \lim_{M \to \infty} \frac{\log N(\epsilon, M)}{M}, \tag{3.1.8}$$

where $N(\epsilon, M)$ is the largest number of ϵ-separated points in X using the metric $d_M : X \times X \to [0, \infty)$ defined by

$$d_M(x,y) = \max_{0 \le m \le M} d(T^m x, T^m y). \tag{3.1.9}$$

Topological entropy is a measure of the exponential growth rate of the number of distinct orbits in F, and a sequence $\xi(n)$ is said to be *deterministic* if it can be realised in a flow of zero-entropy. F is said to be *Möbius disjoint* if all sequences realised in it are asymptotically orthogonal to $\mu(n)$. In this regard, Sarnak's conjecture is

Conjecture 3.1.1 (Sarnak)**.** *If $\xi(n)$ is deterministic then it is of low complexity.*

Conjecture 3.1.1 has been verified in some significant cases. The case in which X is a compact topological group is essentially the work of Davenport [10]. The case in which X is a nilmanifold was verified in Green-Tao [9], and for horocycle flows it was verified in Bourgain-Sarnak-Zeigler [2]. Conjecture 3.1.1 is useful because it guides research to "analyse" the complexity of the Möbius function $\mu(n)$ via the investigation of increasingly complex dynamical systems. For a more detailed account of these definitions and basic concepts see [26] and the references therein.

The motivation of the present chapter is as follows. Sarnak [26] notes that Bourgain (as yet an unpublished result) has proved that the converse of Conjecture 3.1.1 is false. In other words, positive topological entropy does not imply "complexity" in the sense of the negation of Definition 3.1.4. In this direction, some authors focus on constructing positive entropy flows which are not Möbius disjoint. For example, Karagulyan [20] considers sub-shifts of finite type, concluding that all $C^{1+\alpha}$ surface diffeomorphisms with positive entropy are not Möbius disjoint. Another simple example is the left-shift on the orbit closure of the Mobius sequence $\mu(n)$ in $\{-1, 0, 1\}^n$.

3.1.2 Linear transformations of $\mathbb{R}^N$

The objective of this section is to introduce the preliminaries for Theorem 3.2.1. The main tool is Lemma 3.1.14 or, equivalently, Lemma 3.1.18. We assume throughout that $\xi(n)$ is non-trivial and bounded.

Let $V = \mathbb{R}^N$. The sum

$$S(\xi) = \sum_{n \leq N} \mu(n)\xi(n) \tag{3.1.10}$$

may be interpreted as a linear functional (inner product) on V and Definition 3.1.4 may be interpreted as a statement about the angle between the vectors

$$\xi = \sum \xi(m)e_m \tag{3.1.11}$$

and

$$v = \sum \mu(m) e_m \tag{3.1.12}$$

as $N \to \infty$, where $\{e_n\}$ is an orthonormal basis for V with inner product $\langle u, v \rangle = \|u\| \|v\| \cos(u, v)$ and $0 \leq (u, v) < \pi$ is the angle between elements $u, v \in V$. Note that $\|\xi\| \sim CN^{1/2}$ because $\xi(n)$ is non-trivial and bounded. We have

Lemma 3.1.9. *$\xi(n)$ is asymptotically orthogonal to $\mu(n)$ if and only if $\cos(\xi, v) \to 0$ as $N \to \infty$.*

Proof. This is a simple consequence of the fact that $\langle \xi, v \rangle = \|\xi\| \|v\| \cos(\xi, v)$ because

$$\begin{aligned} \|v\| &= \left(\sum_{n \leq N} \mu^2(n) \right)^{1/2} \\ &\sim \frac{(6N)^{1/2}}{\pi}. \end{aligned} \tag{3.1.13}$$

□

Definition 3.1.10. *The set $\{v_n\}$ is the image of the orthonormal basis $\{e_n\}$ under the invertible linear transformation*

$$M^* : e_n \to \sum \mu(m) e_{mn} = v_n. \tag{3.1.14}$$

Since $\xi(m) = \langle \xi, e_m \rangle$, we have

$$\begin{aligned}
\langle \xi, v_n \rangle &= \langle \xi, M^* e_n \rangle \\
&= \langle \xi, \sum \mu(m) e_{mn} \rangle \\
&= \sum \mu(m) \langle \xi, e_{mn} \rangle \\
&= \sum \mu(m) \xi(mn) \qquad (3.1.15)
\end{aligned}$$

and so Lemma 3.1.9 generalises as follows:

Lemma 3.1.11. *$\xi(mn)$ is asymptotically orthogonal to $\mu(m)$ for all n if and only if $\cos(\xi, v_n) \to 0$ as $N \to \infty$ for all n.*

Proof. Same as Lemma 3.1.9, but with $\|v_n\| \sim (6\lfloor N/n \rfloor)^{1/2}/\pi$ and bounded n. If n is unbounded, then $\xi(mn)$ is trivially of low complexity because $\|\xi\| \sim CN^{1/2}$. □

Let M denote the adjoint transformation of M^*, so $\langle \xi, M^* e_n \rangle = \langle M\xi, e_n \rangle$ for all n. As such, low complexity of $\xi(mn)$ is equivalent to the statement that

$$\langle M\xi, e_n \rangle = o(N) \qquad (3.1.16)$$

for all n. The significance of (3.1.16) is that the statement of Lemma 3.1.11 has been translated into a statement about $f = M\xi$.

Lemma 3.1.12. *The adjoint transformation is given by*

$$M : e_n \to \sum_{d|n} \mu\left(\frac{n}{d}\right) e_d, \qquad (3.1.17)$$

and the inverse transformations are

$$M^{-1} : e_n \to \sum_{d|n} e_d \quad and \quad (M^{-1})^* : e_n \to \sum e_{mn}. \tag{3.1.18}$$

Proof. M may be calculated from M^* as follows. For $v \in V$

$$\begin{aligned} M^* v &= \sum \langle v, e_n \rangle \sum \mu(m) e_{mn} = \sum \sum_{n|m} \mu\left(\frac{m}{n}\right) \langle v, e_n \rangle e_m \\ &= \sum \left\langle v, \sum_{n|m} \mu\left(\frac{m}{n}\right) e_n \right\rangle e_m. \end{aligned}$$

The inverse transformations are calculated in a similar way. □

Definition 3.1.13. *Let $\mu_n = Me_n$ and $\mu_n^* = (M^{-1})^* e_n$. Then*

$$\|\mu_n\|^2 = \sum_{d|n} |\mu(d)| = 2^{\omega(n)} \quad and \quad \|\mu_n^*\|^2 = \left\lfloor \frac{N}{n} \right\rfloor. \tag{3.1.19}$$

Also $\{\mu_m, \mu_n^*\}$ is a biorthogonal system, that is

$$\langle \mu_m, \mu_n^* \rangle = \langle Me_m, (M^{-1})^* e_n \rangle = \langle e_m, M^*(M^{-1})^* e_n \rangle = \langle e_m, e_n \rangle = \delta_{mn}$$

so for all $v \in V$

$$v = \sum \langle v, \mu_n^* \rangle \mu_n = \sum \langle v, \mu_n \rangle \mu_n^*. \tag{3.1.20}$$

We now prove a significant property of the norm $\|M\xi\|$:

Lemma 3.1.14. *$\|M\xi\| = o(N)$ if and only if $\xi(mn)$ is asymptotically orthogonal to $\mu(m)$ for every n. If not, then $\|M\xi\| \sim C'N$.*

Proof. Since $\langle M\xi, e_n\rangle = \langle \xi, M^* e_n\rangle = O(N/n)$ we have $\|M\xi\| = O(N)$. Note that

$$\begin{aligned} \|M\xi\|^2 &= \sum \langle M\xi, e_n\rangle^2 \\ &= \sum \langle \xi, v_n\rangle^2 \\ &= \|\xi\|^2 \sum \|v_n\|^2 \cos^2(\xi, v_n) \\ &= \|\xi\|^2 N \sum_{n\le x} \left(N^{-1} \sum_{m\le N/n} \mu^2(m) \right) \cos^2(\xi, v_n) \end{aligned} \tag{3.1.21}$$

and

$$N^{-1} \sum_{m\le N/n} \mu^2(m) \sim \frac{6}{\pi^2 n} \tag{3.1.22}$$

as $N \to \infty$. Since $\|M\xi\| = O(N)$, the series of non-negative terms

$$\frac{\|M\xi\|^2}{N\|\xi\|^2} \sim \frac{\|M\xi\|^2}{CN^2} \sim \frac{6}{\pi^2} \sum_{n\le N} \frac{\cos^2(\xi, v_n)}{n} \tag{3.1.23}$$

is bounded and therefore convergent as $N \to \infty$. By lemma 3.1.11, if $\langle \xi, v_n\rangle = \Omega(N)$ for some $1 \le n \le N$, then $\cos^2(\xi, v_n) > C'$ infinitely often as $N \to \infty$. Supposing such values of n exist, by (3.1.23) we have

$$\frac{\|M\xi\|^2}{N^2} \ge \frac{C''}{n} \tag{3.1.24}$$

for those values of N, so $\|M\xi\| = \Omega(N)$ in this case. On the other hand, if $\langle \xi, v_n\rangle = o(N)$ for every n, then $\lim_{N\to\infty} \cos(\xi, v_n) = 0$ for every $1 \le n \le N$. Since the series (3.1.23) is convergent, it follows that $\|M\xi\| = o(N)$ in this case. □

Remark 3.1.15. We note that M is not a normal transformation. Supposing that M is normal so $\|Mu\| = \|M^*u\|$ for all $u \in V$, we have

$$|\langle \xi, v_n\rangle| = |\langle \xi, M^* e_n\rangle| \le \|\xi\|\|M^* e_n\| = \|\xi\|\|Me_n\| = \|\xi\|\sqrt{2}^{\omega(n)}. \tag{3.1.25}$$

Taking $\xi(m) = \mu(m)$ in (3.1.25) we have $|\langle \xi, v_n \rangle| \sim 6N/\pi^2$ and $\|\xi\| \sim \sqrt{6N}/\pi$, which is clearly false.

Lemma 3.1.16. *For every $v \in V$ one has*

$$\frac{\cos^2(v, e_n)}{n} = \Omega\left(N^{-2}\right) \tag{3.1.26}$$

for at least one $1 \leq n \leq N$ (note that the particular value of n for which this is the case may be different for different values of N).

Proof. It is elementary that

$$\sum_{n \leq N} \cos^2(v, e_n) = 1 \tag{3.1.27}$$

for every $v \in V$. Assuming (3.1.26) is false and summing over n one has a contradiction. □

Corollary 3.1.17. *One has*

$$\frac{\cos^2(M\xi, \mu_n^*)}{n} = \Omega\left(N^{-3}\right) \tag{3.1.28}$$

for at least one $1 \leq n \leq N$.

Proof. Assume that (3.1.28) is false, then we have

$$\cos^2(\xi, e_n) = \frac{\langle M\xi, \mu_n^* \rangle^2}{\|\xi\|^2} = \frac{\|M\xi\|^2 \|\mu_n^*\|^2 \cos^2(M\xi, \mu_n^*)}{\|\xi\|^2} = o\left(N^{-1}\right) \tag{3.1.29}$$

because $\|M\xi\| = O(N)$ and $\|\mu_n^*\| = \lfloor N/n \rfloor$. By (3.1.26) this is clearly false, which proves (3.1.28). □

Lemma 3.1.18. *We have*

$$\frac{\cos^2(M\xi, \mu_n^*)}{n} = O\left(N^{-3}\right) \tag{3.1.30}$$

uniformly for $1 \leq n \leq N$ if and only if $\xi(km)$ correlates with $\mu(m)$ for some k.

Proof. If $\xi(km)$ correlates with $\mu(m)$ for some k then $\|M\xi\| \sim C'N$ by Lemma 3.1.14. Therefore, since $\xi(m)$ is bounded by C'' say, we have

$$\frac{\cos^2(M\xi, \mu_n^*)}{n} = \frac{\xi^2(n)}{n\|M\xi\|^2\|\mu_n^*\|^2} \leq C'''N^{-3}. \tag{3.1.31}$$

On the other hand, if $\xi(km)$ is asymptotically orthogonal to $\mu(m)$ for every k, then

$$\frac{N^2}{\|M\xi\|^2} = g(N) \to \infty \tag{3.1.32}$$

as $N \to \infty$ by Lemma 3.1.14. Thus, supposing that

$$\frac{\cos^2(M\xi, \mu_n^*)}{n} = o\left(g(N)N^{-3}\right) \tag{3.1.33}$$

for all n, we have a contradiction by Corollary 3.1.17. □

3.2 The main Theorem. Concrete examples

We now state and prove Theorem 3.2.1 and offer some concrete examples of Hilbert spaces which may be useful in further work, and which produce explicit examples of sequences which correlate with the Möbius function. Essentially, Theorem 3.2.1 may be viewed as a geometric characterisation of correlation involving only the absolute values $|\cos(M\xi, \mu_n^*)|$, where

$$M : e_n \to \sum_{d|n} \mu\left(\frac{n}{d}\right) e_d = \mu_n \tag{3.2.1}$$

and

$$(M^{-1})^* : e_n \to \sum_{m \le N/n} e_{mn} = \mu_n^*. \tag{3.2.2}$$

Theorem 3.2.1. *Let $\xi(n)$ be bounded and non-trivial, then the following statements are equivalent:*

(a) *$\xi(km)$ correlates with $\mu(m)$ for some k.*

(b) *As $N \to \infty$, we have*

$$\sum_{n \le N} |\cos(M\xi, \mu_n^*)| = O(1). \tag{3.2.3}$$

(c) *As $N \to \infty$, we have*

$$\sum_{n \le N} \cos^2(M\xi, \mu_n^*) = O\left(N^{-1}\right). \tag{3.2.4}$$

Proof. Assume that $\xi(km)$ correlates with $\mu(m)$ for some k. Lemma 3.1.18 shows that $\cos^2(M\xi, \mu_n^*)/n = O(N^{-3})$ uniformly for $1 \le n \le N$, so

$$\sum_{n \le N} |\cos(M\xi, \mu_n^*)| = O\left(N^{-3/2} \sum_{n \le N} n^{1/2}\right) = O(1) \tag{3.2.5}$$

which proves the implication for (b). The implication for (c) is proved similarly.

Now assume that $\xi(km)$ is asymptotically orthogonal to $\mu(m)$ for every k. One has

$$\begin{aligned}
\sum_{n\le N} |\cos(M\xi, \mu_n^*)| &= \frac{1}{\|M\xi\|}\sum_{n\le N}\frac{|\xi(n)|}{\|\mu_n^*\|} \\
&= \frac{N}{\|M\xi\|}\left(N^{-1}\sum_{n\le N}\frac{|\xi(n)|}{\|\mu_n^*\|}\right) \\
&\sim \frac{N}{\|M\xi\|}\left(N^{-3/2}\sum_{n\le N}|\xi(n)|\,n^{1/2}\right) \qquad (3.2.6)
\end{aligned}$$

as $N \to \infty$. Since $\xi(n)$ is bounded and non-trivial, there is a $C > 0$ such that

$$\sum_{n\le N} |\xi(n)| \sim CN \qquad (3.2.7)$$

as $N \to \infty$ and so, by partial summation, one has

$$\begin{aligned}
N^{-3/2}\sum_{n\le N}|\xi(n)|\,n^{1/2} &= N^{-1}\sum_{n\le N}|\xi(n)| - \frac{N^{-3/2}}{2}\int_1^N \sum_{n\le x}|\xi(n)|\,\frac{dx}{x^{1/2}} \\
&\sim C - C/3 \\
&= 2C/3 \qquad (3.2.8)
\end{aligned}$$

as $N \to \infty$. By Lemma 3.1.14 one has $\|M\xi\| = o(N)$, so the r.h.s. of (3.2.6) is unbounded. This proves the converse implication for (*b*). The converse implication for (*c*) is proved similarly. □

3.2.1 Dirichlet series on $\sigma = 1/2 + it$

By Lemma 3.2.1 (or Lemma 3.1.14 directly), when $\xi(n)$ is non-trivial and bounded we may observe that

$$\lim_{N\to\infty} N^{-1}M\xi = \frac{\sqrt{6C}}{\pi}\sum_1^\infty \frac{\cos(\xi, \nu_n)}{n^{1/2}}e_n \qquad (3.2.9)$$

converges in the (real) Hilbert space L^2. The limit is the null element if $\xi(mn)$ is asymptotically orthogonal to $\mu(m)$ for every n but, if $\xi(mn)$ correlates with $\mu(m)$ for some n, then (3.2.9) defines a non-trivial element of L^2.

Choosing as an orthonormal basis the functions $e_n = n^{-it}$, $t \in \mathbb{R}$, the inner product of this space is

$$\langle u, v \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t)\overline{v(t)}dt \tag{3.2.10}$$

and the dual is the space of square summable real sequences ℓ^2. In this case, we may write

$$\Xi(t) = \sum_{1}^{\infty} \frac{\cos(\xi, \nu_n)}{n^{1/2+it}} \tag{3.2.11}$$

and ask the following general question: can properties of the arithmetic sequence $\xi(n)$ be determined by analytic properties of the Dirichlet series $\Xi(t)$?

Example 3.2.2. A somewhat tautological example occurs when $\xi(n) = \mu(n)$, in which case

$$\Xi(t) = \frac{6}{\pi^2} \prod_{p} \left(1 - \frac{p^{-3/2-it}}{1+p^{-1}}\right). \tag{3.2.12}$$

3.2.2 Polynomials on the unit circle

Another concrete example of $\mathbb{R}^N$ is the space of polynomials of degree N with real coefficients and zero constant term. Corollary 3.2.3 below demonstrates a connection between low complexity and the values taken by this set of polynomials on the unit circle.

Corollary 3.2.3. *Let* $\{a_m\}_{m\le N}$ *be real and*

$$f(z) = \sum_{m\le N} a_m z^m. \tag{3.2.13}$$

Let ω_m *be an* m*th root of unity. If the sequence*

$$\xi(m) = \frac{1}{m}\sum_{1}^{m} f(\omega_m^k) \tag{3.2.14}$$

is non-trivial and bounded, then

$$\|f\| = \left(\sum_{m\le N} a_m^2\right)^{1/2} = o(N) \tag{3.2.15}$$

if and only if $\xi(mn)$ *is of low complexity for every* n*.*

Proof. As a Hilbert space of dimension N over the field $\mathbb{R}$, Lemma 3.1.14 is applicable. Thus, assuming that $\xi(m)$ is non-trivial and bounded, one is required to show that (3.2.14) implies $f = M\xi$.

Let γ be a path enclosing the unit disc, then we have

$$\begin{aligned}
\xi(m) = \frac{1}{m}\sum_{1}^{m} f(\omega_m^k) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{(z^m-1)z} \\
&= \frac{1}{2\pi i}\sum_{n\le N/m}\int_\gamma \frac{f(z)dz}{z^{mn+1}} \\
&= \sum_{n\le N/m} a_{mn}
\end{aligned} \tag{3.2.16}$$

and so, by Mobius inversion on the r.h.s of (3.2.16), we have

$$a_n = \sum_{m\le N/n} \mu(m)\xi(mn), \tag{3.2.17}$$

which completes the proof. □

Remark 3.2.4. Corolary 3.2.3 reveals a curious duality between non-trivial bounded real sequences of low complexity and sequences of polynomials of degree $N \to \infty$ with coefficients

$$a_n = \sum_{m \le N/n} \mu(m)\xi(mn). \tag{3.2.18}$$

Indeed, corollary 3.2.3 shows that $a_n = o(N)$ for all n if and only if $\xi(mn)$ is of low complexity for all n. However, since $\xi(m)$ is non-trivial and real, the coefficients a_n must be oscillatory (the presence of $\mu(m)$ causing significant cancellation). Presumably then, each a_n behaves like an aggregate of Bernoulli trails. In other words, f is essentially a random polynomial.

This duality is of an inverse nature: such polynomials are essentially random, yet Corollary 3.2.3 shows that $\xi(mn)$ is of low complexity for every n in this case. In other words, *if the polynomial is random, then its averages on the unit circle are structured in the Möbius sense*. A potential explanation for this phenomenon is that *random polynomials tend to have roots distributed uniformly close to the unit circle*, so the averages on the unit circle (3.2.14) are structured in some sense. It would be interesting to investigate this quantitatively, particularly in view of the apparent structure in the Möbius sense, and there do exist results in this direction (for example Hughes and Nikeghbali [16]).

Bibliography

[1] J. C. Andrade, L. Bary-Soroker, Z. Rudnick, *Shifted convolution and the Titchmarsh divisor problem over* $F_q[t]$, Number fields and function fields: coalescences, contrasts and emerging applications. Phil. Trans. R. Soc. A 373: 20140315. (2015).

[2] J. Bourgain, P. Sarnak, T. Ziegler, *Distjointness of Mobius from horocycle flows*, arXiv:1110.0992.

[3] J.B. Conrey, S.M. Gonek, *High moments of the Riemann zeta-function*, Duke Math. J. 107 (2001), 577-604.

[4] H. Delange, *Sur les formules dues á Atle Selberg*, Bull. Sc. Math. 2 série 83 (1959), 101-111.

[5] W. Duke, J.B. Friedlander, and H. Iwaniec, *A quadratic divisor problem*, Invent. Math. 115 (1994), no. 2, 209-217.

[6] T. Estermann, *Uber die Darstellung einer Zahl als Differenz von zwei Produkten*, J. Reine Angew. Math. 164 (1931), 173-182.

[7] P. Erdős. and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, Amer. J. Math 62, (1940) 738-742.

[8] P. Erdős, R.R. Hall *Values of the divisor function on short intervals*, J. Number Theory 12 (1980), 176-187.

[9] B. Green, T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. 175 (2012), 541-566.

[10] H. Davenport, *On some infinite series involving arithmetical functions (II)* Quart. J. Math., 8 (1937), pp. 313-320.

[11] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1-49.

[12] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*, Quart. J. Math. 48 (1917), 76-92.

[13] R. R. Hall, G. Tenenbaum, *On the local behaviour of some arithmetical functions*, Acta Arith. 43 (1984), no. 4, 375-390.

[14] D.R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. London Math. Soc. (3) 38 (1979), no. 3, 385-422.

[15] K. Henriot, *Nair-Tenenbaum bounds uniform with respect to the discriminant*, Math Proceedings of the Cambridge Philosophical Society 153 (2012), no. 3, 405-424.

[16] C. Hughes and A. Nikeghbali, *The zeros of random polynomials cluster uniformly near the unit circle*, Compositio Mathematica, 2008, 144, 734- 746.

[17] A. Ingham, *Some asymptotic formulae in the theory of numbers*, J. London Math. Soc. 2 (1927), 202-208.

[18] A. Ivić, *The general additive divisor problem and moments of the zeta-function*, New trends in probability and statistics, Vol. 4 (Palanga, 1996), 69-89, VSP, Utrecht, 1997.

[19] A. Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 205-243, London Math. Soc. Lecture Note Ser., 237, Cambridge Univ. Press, Cambridge, 1997.

[20] Karagulyan, *On Moebius orthogonality for subshifts of finite type with positive topological entropy*, arXiv:1701.01968.

[21] I. Kátai, *On the local behaviour of the function* $d(n)$ (Hungarian.) Mat. Lapok 18 (1967) 297-302.

[22] E. Landau, *Uber einen Satz von Tschebyschef*, Math. Annalen, 61 (1905), 527-550.

[23] Y. Motohashi, *The binary additive divisor problem*, Ann. Sci École Norm. Sup. (4)27 (1994) 529-572

[24] M. Nair, G. Tenenbaum, *Short sums of certain arithmetic functions*, Acta Math. 180 (1998), no. 1, 119-144.

[25] N. Ng, M. Thom, *Bounds and conjectures for additive divisor sums*, arXiv:1609.01411v1 (2016).

[26] P. Sarnak, *Three lectures on the Möbius function, randomness and dynamics*, http://publications.ias.edu/sarnak/.

[27] A. Selberg, *Note on the paper by L. G. Sathe*, J. Indian Math. Soc. 18 (1954), 83-87.

[28] T. Tao, *Heuristic computation of correlations of higher order divisor functions*, https: //terrytao.wordpress.com.

[29] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed. Oxford Univ. Press, New York, 1986.

[30] C.J. de la Vallée Poussin, *Recherches analytiques sur la théorie des nombres premiers*, Ann. Soc. Sci. Bruxelles 20 (1896), 183-256.